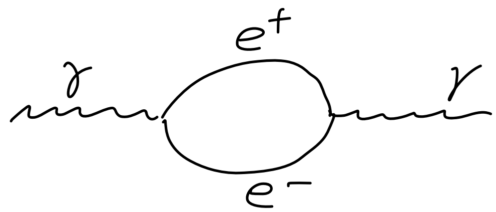
§ 4.5 Polarizing the Vacuum and Renormalizing the Charge

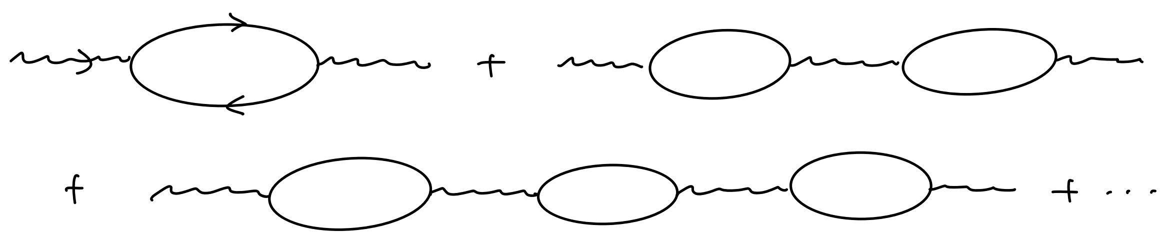
Pair production and annihilation

Quantum fluctuations can turn a photon into an electron and a positron and vice versa:

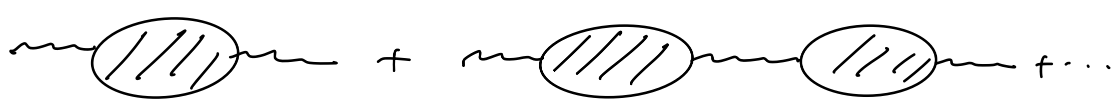
$$\gamma \longrightarrow e^+ e^- \quad , \quad e^+ e^- \longrightarrow \gamma$$



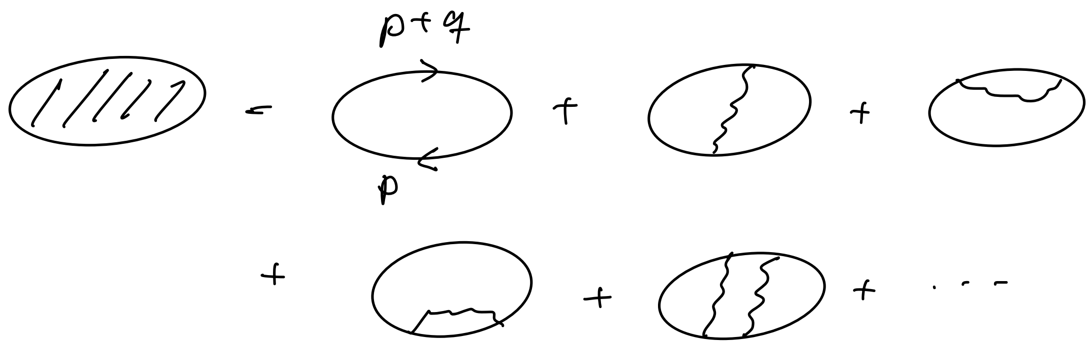
This process can repeat itself:



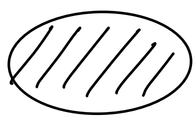
There are furthe quantum fluctuations giving altogether:



where



We define



$$=: i\Pi_{\mu\nu}(q)$$

It is convenient to rewrite the Lagrangian

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

by letting $A \mapsto \frac{1}{e}A$, giving

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu(\partial_\mu - iA_\mu) - m\right]\psi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}$$

$\rightarrow$ the gauge trf. is now written as

$$\psi \longmapsto e^{i\alpha}\psi, \qquad A_\mu \longmapsto A_\mu + \partial_\mu \alpha$$

and the photon propagator is given by

$$iD_{\mu\nu}(q) = \frac{-ie^2}{q^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]$$

The diagrammatic proof of gauge invariance given in chapter §3.5 implies

$$q^\mu \Pi_{\mu\nu}(q) = 0 \qquad (*)$$

$\rightarrow$ together with Lorentz invariance:

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - \eta_{\mu\nu} q^2)\Pi(q^2)$$

$\rightarrow$ the renormalized photon propagator is then given by the geometric series

$$iD^{P}_{\mu\nu}(q) = iD_{\mu\nu}(q) + iD_\mu{}^\lambda(q) i\Pi_{\lambda\rho}(q) iD^\rho{}_\nu(q)$$
$$+ iD_\mu{}^\lambda(q) i\Pi_{\lambda\rho} iD^\rho{}_\sigma(q) i\Pi_{\sigma\kappa}(q) iD^\kappa{}_\nu(q) + \dots$$
$$= \frac{-ie^2}{q^2}\eta_{\mu\nu}\left(1 - e^2\Pi(q^2) + [e^2\Pi(q^2)]^2 + \dots\right)$$
$$+ q_\mu q_\nu \text{ term}$$

Because of $(*)$, the $(1-\xi)\frac{q_\mu q_\nu}{q^2}$ part of of $D_{\mu\nu}(q)$ is annihilated when it encounters $\Pi_{\lambda\rho}(q)$.

$\rightarrow$ residue of pole: $\quad e_R^2 = e^2 \dfrac{1}{1+e^2\Pi(0)}$

Gauge invariance

In order to determine e_R, we calculate to lowest order

$$i\Pi_{\mu\nu}(q) = (-)\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(i\gamma^\nu \frac{i}{\not{p}+\not{q}-m}\, i\gamma^\mu \frac{i}{\not{p}-m}\right) \quad (1)$$

→ for large p, integrand goes as $\frac{1}{p^2}$ with subleading term $\frac{m^2}{p^4}$

→ quadratic and logarithmic divergences

In Pauli-Villars regularization, we replace

$$i\Pi_{\mu\nu}(q) = (-)\int \frac{d^4p}{(2\pi)^4}\left[\mathrm{tr}\left(i\gamma^\nu \frac{i}{\not{p}+\not{q}-m}\, i\gamma^\mu \frac{i}{\not{p}-m}\right) - \sum_a c_a\, \mathrm{tr}\left(i\gamma^\nu \frac{i}{\not{p}+\not{q}-m_a}\, i\gamma^\mu \frac{i}{\not{p}-m_a}\right)\right.$$

→ integrand goes as

$$\left(1-\sum_a c_a\right)\frac{1}{p^2} \quad \text{and} \quad \left(m^2-\sum_a c_a m_a^2\right)\frac{1}{p^4}$$

→ fix c_a and m_a such that

$$\sum_a c_a = 1, \quad \sum_a c_a m_a^2 = m^2 \quad (2)$$

→ introduce two regulator masses

Expanding in powers of q, we have

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - \eta_{\mu\nu} q^2)[\Pi(0) + \dots]$$

→ we are only interested in terms of order $\mathcal{O}(q^2)$ and higher in the Feynman integral.

→ expanding (1), we see that the of order $\mathcal{O}(q^2)$ term goes as $\frac{1}{p^4}$ giving logarithmically divergent contribution

→ only need one regulator

Using

$$\frac{1}{\not{p} + \not{q} - m} \not{q} \frac{1}{\not{p} - m} = \frac{1}{\not{p} - m} - \frac{1}{\not{p} + \not{q} - m}$$

we see that the two pieces cancel upon shifting integration variable $p \to p+q$ (allowed once using regulator to make integral convergent)

→ after few steps (1) gives

$$i\Pi_{\mu\nu}(q) = -i\int \frac{d^4p}{(2\pi)^4}\,\frac{N_{\mu\nu}}{D}$$

where $N_{\mu\nu} = tr[\gamma_\nu(\not p + \not q + m]\gamma_\mu(\not p - m)]$ and

$$\frac{1}{D} = \int_0^1 d\alpha\,\frac{1}{\mathcal{D}}$$

with $\mathcal{D} = [\ell^2 + \alpha(1-\alpha)q^2 - m^2 + i\varepsilon]^2$, where

$$\ell = p + \alpha q$$

$$N_{\mu\nu} \longrightarrow$$

$$-4\left(\frac{1}{2}\eta_{\mu\nu}\ell^2 + \alpha(1-\alpha)(2q_\mu q_\nu - \eta_{\mu\nu}q^2) - m^2\eta_{\mu\nu}\right)$$

Adding contribution of regulators and integration over ℓ gives

(3) $$\Pi_{\mu\nu}(q) = -\frac{1}{4\pi^2}\int_0^1 d\alpha\left[F_{\mu\nu}(m) - \sum_a c_a F_{\mu\nu}(m_a)\right]$$

where

$$F_{\mu\nu}(m) = \frac{1}{2}\eta_{\mu\nu}\left(\Lambda^2 - 2[m^2 - \alpha(1-\alpha)q^2]\log\frac{\Lambda^2}{m^2-\alpha(1-\alpha)q^2} + m^2 - \alpha(1-\alpha)q^2\right)$$

$$-\left[\alpha(1-\alpha)(2q_\mu q_\nu - \eta_{\mu\nu} q^2) + m^2\eta_{\mu\nu}\right]\left[\log\frac{\Lambda^2}{m^2-\alpha(1-\alpha)q^2} - 1\right]$$

(2) $\longrightarrow$ contribution of Λ drops out in computation of (3)

Finally, combining every thing, we get

$$\Pi_{\mu\nu}(q) = -\frac{1}{2\pi^2}(q_\mu q_\nu - \eta_{\mu\nu} q^2) \int_0^1 d\alpha\, \alpha(1-\alpha)$$

$$\times\left(\log\left[m^2-\alpha(1-\alpha)q^2\right] - \sum_a c_a \log\left[m_a^2-\alpha(1-\alpha)q^2\right]\right) \quad (4)$$

$\longrightarrow$ we see that indeed the factor $(q_\mu q_\nu - \eta_{\mu\nu} q^2)$ pops out!

For $q^2 \ll m_a$, we define

$$M^2 := \sum_a c_a \log m_a^2$$

giving

$$\Pi(q^2) = \frac{1}{2\pi^2}\int_0^1 d\alpha\,(1-\alpha)\log\frac{M^2}{m^2-\alpha(1-\alpha)q^2}$$

$\longrightarrow$ effectively need one regular as anticipated!

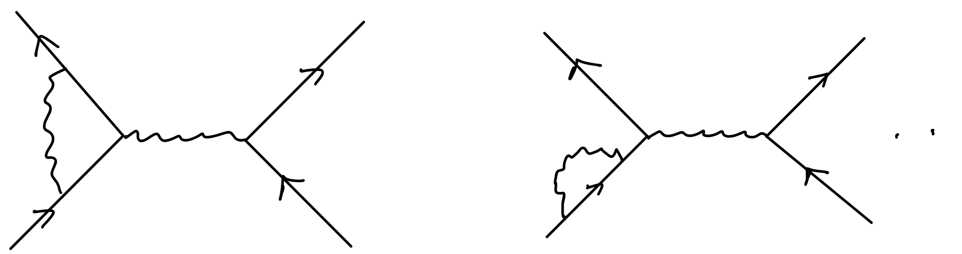
Electric charge

We conclude that

$$e_R^2 = e^2 \frac{1}{1 + (e^2/12\pi^2)\log(M^2/m^2)}$$

$$\simeq e^2 \left(1 - \frac{e^2}{12\pi^2} \log\left(\frac{M^2}{m^2}\right)\right)$$

How about diagrams



Using our new Lagrangian with covariant derivative $D_\mu = \partial_\mu - iA_\mu$, we see that they don't contribute to charge renormalization!